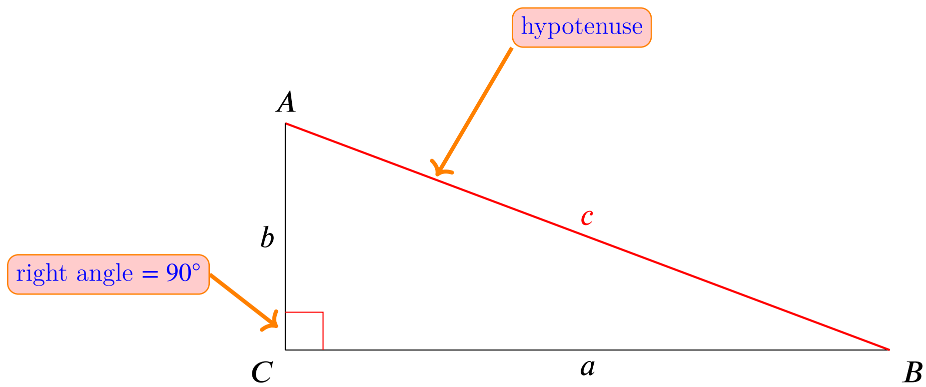
4.4. Pythagorean Theorem

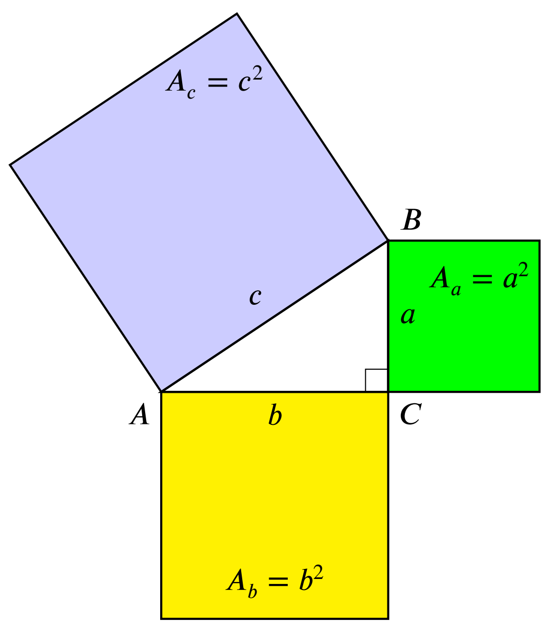
A **right-angle triangle** is a triangle containing a right angle (90°). A triangle cannot have more than one right angle, since the sum of the two right angles plus the third angle would exceed the 180° total possessed by a triangle. The side opposite the right angle is called the hypotenuse (side c in the figure below). The sides adjacent to the right angle are called legs (or catheti, singular: cathetus).



Math notation for right-angle triangle

The Pythagorean Theorem states that the square of a hypotenuse is equal to the sum of the squares of the other two sides. It is one of the fundamental relations in Euclidean geometry.

$$c^2 = a^2 + b^2$$

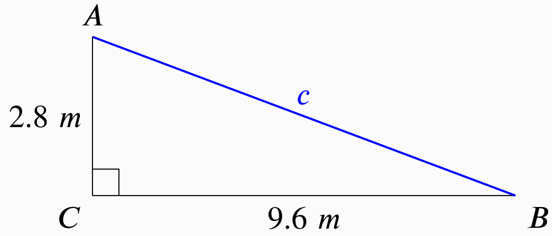


Pythagorean triangle with the squares of its sides and labels

Pythagorean triples are integer values of a, b, c satisfying this equation.

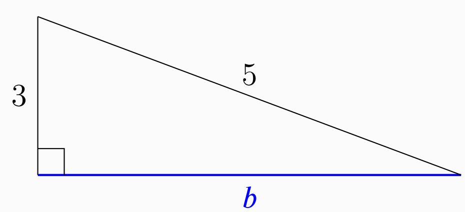
Finding the Sides of a Right Angled Triangle

Example 1: Find the hypotenuse.



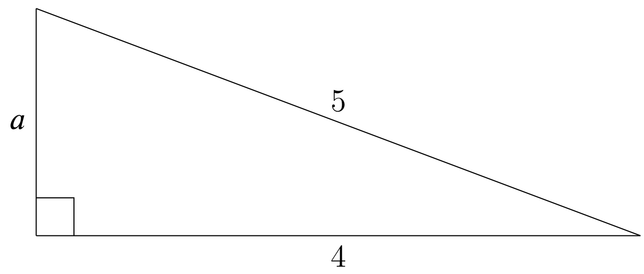
$$
\begin{aligned}
c^2 &= a^2 + b^2 && |\text{substitute for } a \text{ and } b\\
c^2 &= (9.6\ m)^2 + (2.8\ m)^2\\
c^2 &= 92.16\ m^2 + 7.84\ m^2\\
c^2 &= 100\ m^2 && |\text{take the square root of each side}\\
\sqrt{c^2} &= \sqrt{100\ m^2}\\
\sqrt{c^2} &= \sqrt{100}\sqrt{m^2}\\
c &= 10\ m
\end{aligned}
$$

Example 2: Find the missing side b.

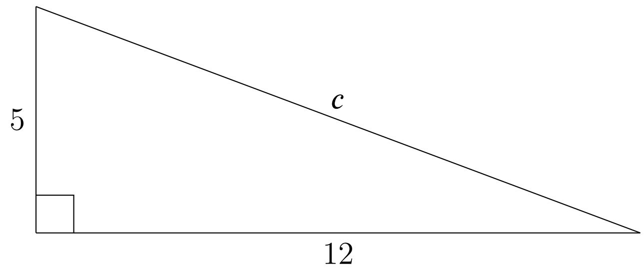


$$
\begin{aligned}
c^2 &= a^2 + b^2 && |\text{subtract } a^2 \text{ from each side}\\
c^2 - a^2 &= \cancel{a^2} + b^2 \cancel{- a^2}\\
c^2 - a^2 &= b^2 && |\text{switch sides}\\
b^2 &= c^2 - a^2 && |\text{substitute for } a \text{ and } c\\
b^2 &= 5^2 - 3^2\\
b^2 &= 25 - 9 = 16 && |\text{take the square root of each side}\\
\sqrt{b^2} &= \sqrt{16}\\
b &= 4
\end{aligned}
$$

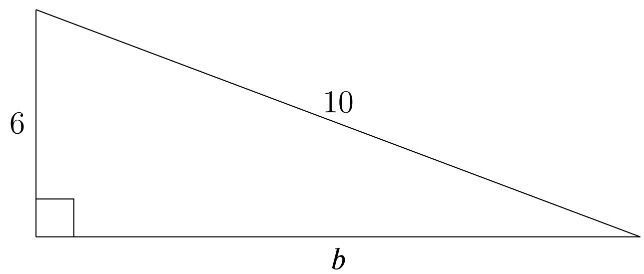
Practice 1: Find all the missing sides in each right angled triangle.



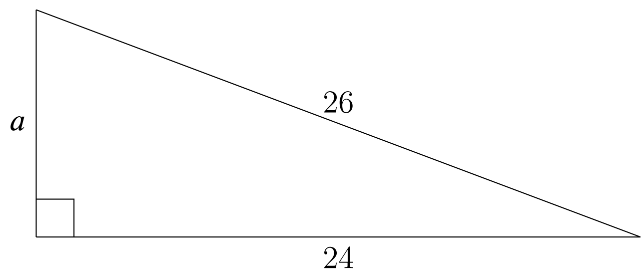
$$c^2 = a^2 + b^2$$
$$a^2 = c^2 - b^2$$
$$a^2 = 5^2 - 4^2 = 25 - 16 = 9$$
$$\sqrt{a^2} = \sqrt{9}$$
$$a = 3$$



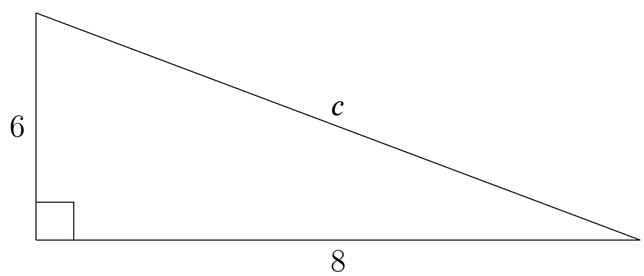
$$c^2 = a^2 + b^2$$
$$c^2 = 5^2 + 12^2 = 25 + 144 = 169$$
$$\sqrt{c^2} = \sqrt{169}$$
$$c = 13$$



$$c^2 = a^2 + b^2$$
$$b^2 = c^2 - a^2$$
$$b^2 = 10^2 - 6^2 = 100 - 36 = 64$$
$$\sqrt{b^2} = \sqrt{64}$$
$$b = 8$$

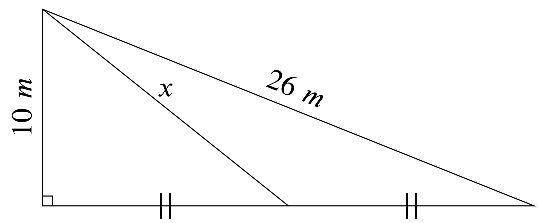


$$c^2 = a^2 + b^2$$
$$a^2 = c^2 - b^2$$
$$a^2 = 26^2 - 24^2 = 676 - 576 = 100$$
$$\sqrt{a^2} = \sqrt{100}$$
$$a = 10$$



$$c^2 = a^2 + b^2$$
$$c^2 = 6^2 + 8^2 = 36 + 64 = 100$$
$$\sqrt{c^2} = \sqrt{100}$$
$$c = 10$$

Challenge 1: Find x.



Use the Pythagorean theorem to find the base b in the larger triangle:

Given: hypotenuse : $c = 26\ m$ $\quad a = 10\ m$ $\quad$ Find: $b = ?$

$$a^2 + b^2 = c^2 \qquad |\text{subtract } a^2 \text{ from both sides}$$

$$b^2 = (26\ m)^2 - (10\ m)^2 \qquad |\text{substitute values } c = 26\ m \text{ and } a = 10\ m$$

$$b^2 = 676\ m^2 - 10\ m^2$$

$$b^2 = 576\ m^2$$

$$\sqrt{b^2} = \sqrt{576\ m^2} \qquad |\text{take the square root of each side}$$

$$b = 24\ m$$

In the smaller triangle, the base (let's call it y) is half the base of the larger triangle.

Given: $y = \frac{1}{2}b = \frac{1}{2}(24\ m) = 12\ m$ $\quad a = 10\ m$ $\quad$ Find: hypotenuse : $x = ?$

Use the Pythagorean theorem:

$$c^2 = a^2 + b^2 \qquad |\text{substitute: } c = x,\ a = a \text{ and } b = y$$

$$x^2 = a^2 + y^2 \qquad |\text{substitute values: } a = 10\ m \text{ and } y = 12\ cm$$

$$x^2 = (10\ m)^2 + (12\ m)^2$$

$$x^2 = 100\ m^2 + 144\ m^2$$

$$x^2 = 244\ m^2 \qquad |\text{take the square root of each side}$$

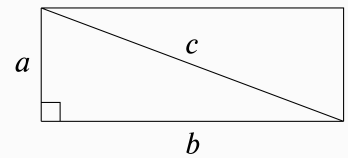
$$\sqrt{x^2} = \sqrt{244\ m^2}$$

$$x = 15.6\ m$$

Area of Right Angled Triangle

The area of a triangle is equal to one half the base multiplied by the corresponding height: $A = \dfrac{bh}{2}$

Example 3: Find the length of the diagonal of a rectangle that has width $a = 3$ and length $b = 4$.

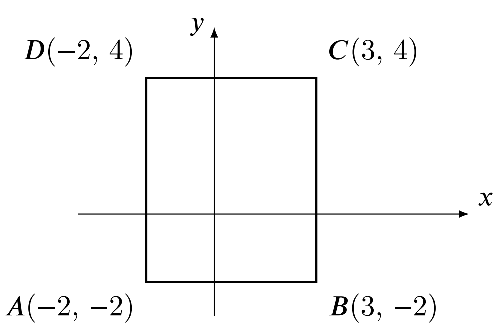


The diagonal of a rectangle is the hypotenuse of a right-angle triangle. Use the Pythagorean Theorem.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ c^2 &= 3^2 + 4^2 = 9 + 16 = 25 \\ \sqrt{c^2} &= \sqrt{25} \\ c &= 5 \end{aligned}$$

The length of the diagonal is $c = 5$.

Challenge 2: What is the length BD shown in the figure?



Points A, B, and D form the right-angle triangle with hypotenuse BD.

The length AB equals to change in x, from -2 in A($\underline{-2}$, -2) to 3 in B($\underline{3}$, -2), since points A and B have a fixed y coordinate.

$$AB = \Delta x = 3 - (-2) = 3 + 2 = 5$$

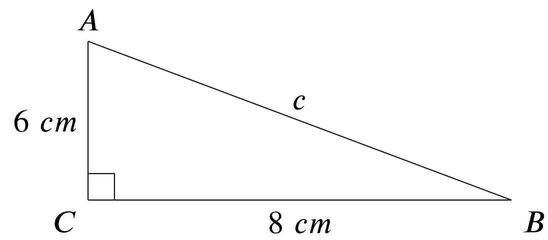
The length AD equals to change in y, from -2 in A(-2, $\underline{-2}$) to 4 in B(-2, $\underline{4}$), since points A and B have a fixed x coordinate.

$$AD = \Delta y = 4 - (-2) = 4 + 2 = 6$$

Use the Pythagorean theorem to find length BD, which is a hypotenuse of the right-angle triangle BAD.

$$\begin{aligned} BD^2 &= 5^2 + 6^2 = 25 + 36 \\ BD^2 &= 61 \\ \sqrt{BD^2} &= \sqrt{61} \\ BD &\approx 7.8 \end{aligned}$$

Challenge 3: Find the area of a square whose perimeter is the same as the perimeter of the triangle shown below.



To find the perimeter of the triangle we need to find the hypotenuse, side c, first.

$$P = a + b + c$$

The hypotenuse c of the triangle is

$$\begin{aligned} c^2 &= (6\ cm)^2 + (8\ cm)^2 \\ &= 36\ cm^2 + 64\ cm^2 \\ &= 100\ cm^2 \end{aligned}$$

Take square root of both sides

$$\begin{aligned} \sqrt{c^2} &= \sqrt{100\ cm^2} \\ c &= 10\ cm \end{aligned}$$

The perimeter of triangle P is the sum of the lengths of all three sides of the triangle:

$$P = 6\ cm + 8\ cm + 10\ cm = 24\ cm$$

Perimeters of the triangle and the square are same. If the side of the square is a, then the perimeter of the square is

$$P = 4a$$

Divide by 4 and switch sides

$$a = \frac{P}{4}$$

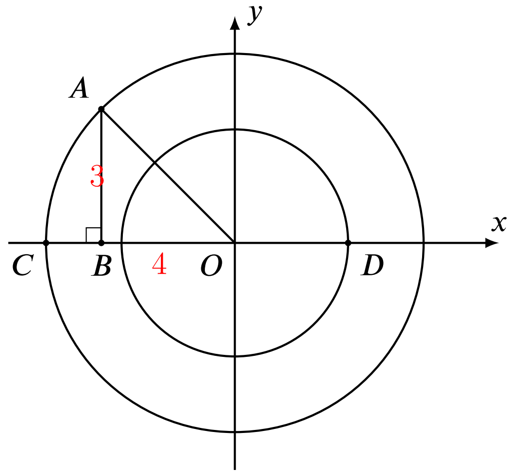
Substitute value for P

$$a = \frac{24\ cm}{4} = 6\ cm$$

Finally, the area of the square is

$$A_S = a^2 = (6\ cm)^2 = 36\ cm^2$$

Challenge 4: Two concentric circles with the center at the origin are shown below. $A(4,\ 3)$ is on the larger circle and CD is 9, what is the radius of the smaller circle? 😐



The radius, r_0 of the larger circle is distance AO. To find this distance we use x and y coordinates of point A (right angled triangle $\triangle ABO$. Using Pythagorean theorem we can find the radius of the larger circle, r_0:

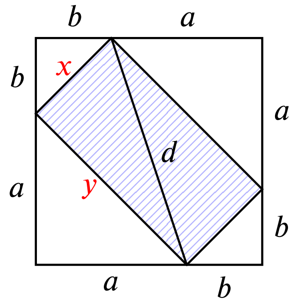
$$r_0^2 = 4^2 + 3^2$$
$$r_0 = \sqrt{16+9} = \sqrt{25} = 5$$

Line segment CD is split into CO and OD. Furthermore, $CD = 9$, and CO is the radius of the larger circle, and OD is the radius of the smaller circle

$$CD = CO + OD$$
$$9 = 5 + OD$$
$$OD = 4$$

The radius of the smaller circle is 4.

Challenge 5: A rectangle with the area of $3\sqrt{2}$ is inscribed into a square with the side length of $a+b$, as shown below. Find the length of the rectangles' diagonal d, if $a : b = 2\sqrt{2} : 3$.



Let x and y be the height and the width of the rectangle, respectively. There are four isosceles triangles around the rectangle: two smaller triangles with legs of length b (top left and bottom right corners), and two larger triangles with legs of length a (top right and bottom left corners). Using the Pythagorean Theorem we can write:

$$x^2 = b^2 + b^2 \qquad y^2 = a^2 + a^2$$
$$x^2 = 2b^2 \qquad y^2 = 2a^2$$
$$x = \sqrt{2b^2} = b\sqrt{2} \qquad y = \sqrt{2a^2} = a\sqrt{2}$$

The area of the rectangle is:

$$A_R = x \cdot y \qquad |\text{substitute values } x = b\sqrt{2} \text{ and } y = a\sqrt{2}$$
$$A_R = b\sqrt{2} \cdot a\sqrt{2}$$
$$A_R = 2ab \qquad |\text{using the ratio } \frac{a}{b} = \frac{2\sqrt{2}}{3}\text{, we can write } a = \frac{2\sqrt{2}}{3} \cdot b$$
$$A_R = 2 \cdot \left(\frac{2\sqrt{2}}{3} \cdot b\right) \cdot b \qquad |\text{simplify and substitute the value } A_R = 3\sqrt{2}$$
$$3\sqrt{2} = \frac{4\sqrt{2}}{3} b^2 \qquad | \cdot \frac{4\sqrt{2}}{3}$$
$$3\sqrt{2} \cdot \frac{3}{4\sqrt{2}} = b^2$$
$$b^2 = \frac{9}{4}$$
$$b = \frac{3}{2}$$

Knowing b we can calculate a by using the ratio: $a = \frac{2\sqrt{2}}{3} \cdot b = \frac{2\sqrt{2}}{3} \cdot \frac{3}{2} = \sqrt{2}$. We can now proceed by calculating $x = b\sqrt{2} = \frac{3}{2} \cdot \sqrt{2} = \frac{3\sqrt{2}}{2}$ and $y = a\sqrt{2} = \sqrt{2} \cdot \sqrt{2} = 2$, and finally calculate the diagonal using the Pythagorean theorem:

$$d^2 = x^2 + y^2$$
$$d = \sqrt{\left(\frac{3\sqrt{2}}{2}\right)^2 + 2^2} = \sqrt{\frac{9 \cdot 2}{4} + 4} = \sqrt{\frac{9}{2} + \frac{8}{2}} = \sqrt{\frac{17}{2}} = \frac{\sqrt{17}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{17}\sqrt{2}}{\left(\sqrt{2}\right)^2} = \frac{\sqrt{34}}{2}$$