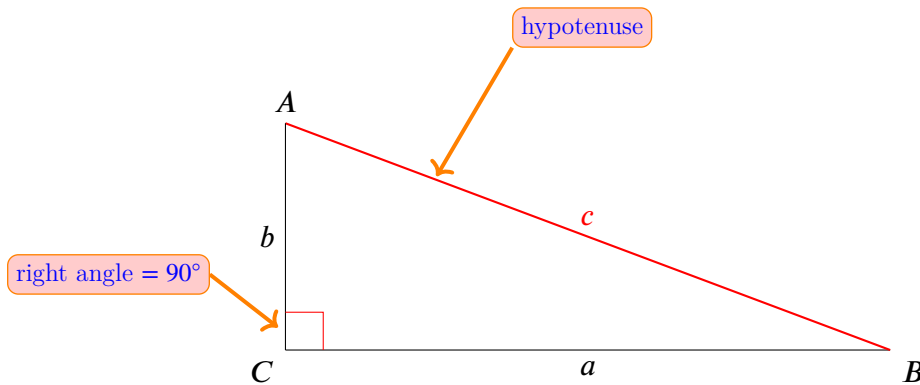


4.4. Pythagorean Theorem

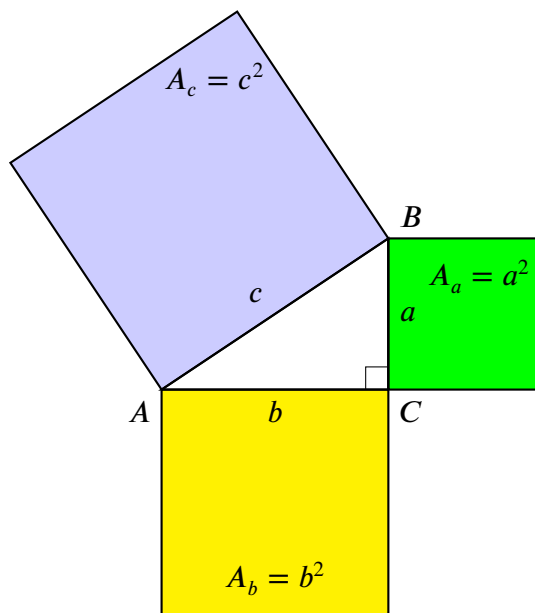
A **right-angle triangle** is a triangle containing a right angle (90°). A triangle cannot have more than one right angle, since the sum of the two right angles plus the third angle would exceed the 180° total possessed by a triangle. The side opposite the right angle is called the hypotenuse (side c in the figure below). The sides adjacent to the right angle are called legs (or catheti, singular: cathetus).



Math notation for right-angle triangle

The Pythagorean Theorem states that the square of a hypotenuse is equal to the sum of the squares of the other two sides. It is one of the fundamental relations in Euclidean geometry.

$$c^2 = a^2 + b^2$$

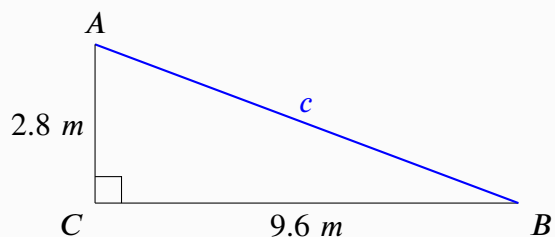


Pythagorean triangle with the squares of its sides and labels

Pythagorean triples are integer values of a , b , c satisfying this equation.

Finding the Sides of a Right Angled Triangle

Example 1: Find the hypotenuse. 🏠



$$c^2 = a^2 + b^2 \quad \text{|substitute for } a \text{ and } b$$

$$c^2 = (9.6 \text{ m})^2 + (2.8 \text{ m})^2$$

$$c^2 = 92.16 \text{ m}^2 + 7.84 \text{ m}^2$$

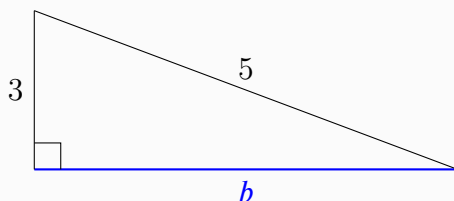
$$c^2 = 100 \text{ m}^2 \quad \text{|take the square root of each side}$$

$$\sqrt{c^2} = \sqrt{100 \text{ m}^2}$$

$$\sqrt{c^2} = \sqrt{100} \sqrt{\text{m}^2}$$

$$c = 10 \text{ m}$$

Example 2: Find the missing side b .



$$c^2 = a^2 + b^2 \quad \text{|subtract } a^2 \text{ from each side}$$

$$c^2 - a^2 = \cancel{a^2} + b^2 - \cancel{a^2}$$

$$c^2 - a^2 = b^2 \quad \text{|switch sides}$$

$$b^2 = c^2 - a^2 \quad \text{|substitute for } a \text{ and } c$$

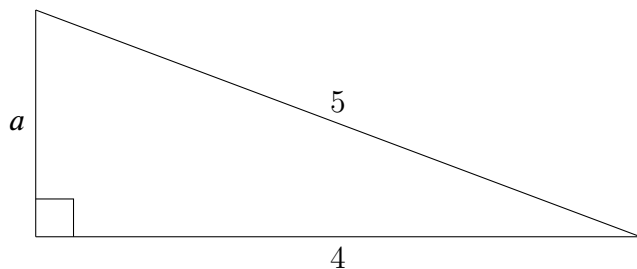
$$b^2 = 5^2 - 3^2$$

$$b^2 = 25 - 9 = 16 \quad \text{|take the square root of each side}$$

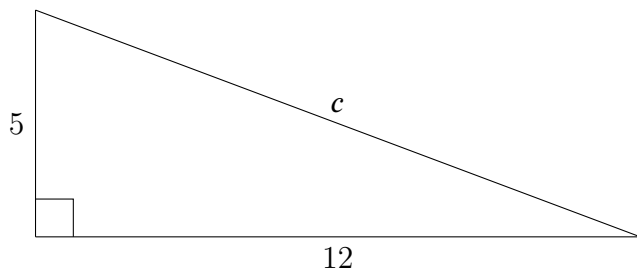
$$\sqrt{b^2} = \sqrt{16}$$

$$b = 4$$

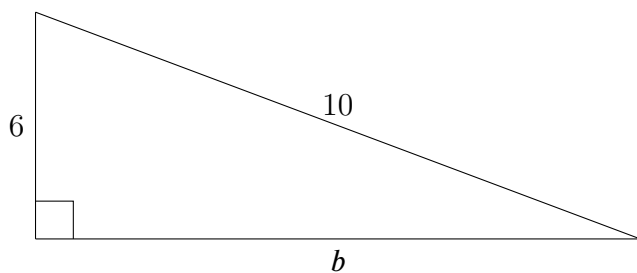
Practice 1: Find all the missing sides in each right angled triangle.



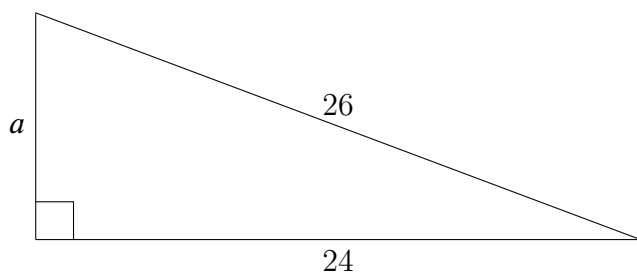
$$\begin{aligned}c^2 &= a^2 + b^2 \\a^2 &= c^2 - b^2 \\a^2 &= 5^2 - 4^2 = 25 - 16 = 9 \\\sqrt{a^2} &= \sqrt{9} \\a &= 3\end{aligned}$$



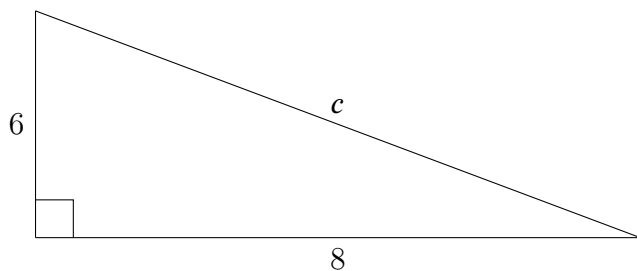
$$\begin{aligned}c^2 &= a^2 + b^2 \\c^2 &= 5^2 + 12^2 = 25 + 144 = 169 \\\sqrt{c^2} &= \sqrt{169} \\c &= 13\end{aligned}$$



$$\begin{aligned}c^2 &= a^2 + b^2 \\b^2 &= c^2 - a^2 \\b^2 &= 10^2 - 6^2 = 100 - 36 = 64 \\\sqrt{b^2} &= \sqrt{64} \\b &= 8\end{aligned}$$

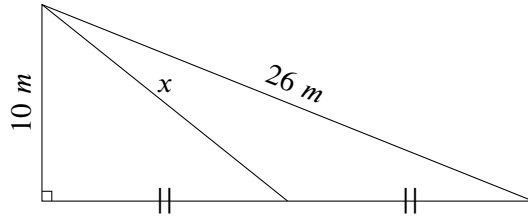


$$\begin{aligned}c^2 &= a^2 + b^2 \\a^2 &= c^2 - b^2 \\a^2 &= 26^2 - 24^2 = 676 - 576 = 100 \\\sqrt{a^2} &= \sqrt{100} \\a &= 10\end{aligned}$$



$$\begin{aligned}c^2 &= a^2 + b^2 \\c^2 &= 6^2 + 8^2 = 36 + 64 = 100 \\\sqrt{c^2} &= \sqrt{100} \\c &= 10\end{aligned}$$

Challenge 1: Find x . 🧮 😊



Use the Pythagorean theorem to find the base b in the larger triangle:

Given: hypotenuse : $c = 26\text{ m}$ $a = 10\text{ m}$ Find: $b = ?$

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{|subtract } a^2 \text{ from both sides} \\ b^2 &= (26\text{ m})^2 - (10\text{ m})^2 && \text{|substitute values } c = 26\text{ m} \text{ and } a = 10\text{ m} \\ b^2 &= 676\text{ m}^2 - 100\text{ m}^2 \\ b^2 &= 576\text{ m}^2 \\ \sqrt{b^2} &= \sqrt{576\text{ m}^2} && \text{|take the square root of each side} \\ b &= 24\text{ m} \end{aligned}$$

In the smaller triangle, the base (let's call it y) is half the base of the larger triangle.

Given: $y = \frac{1}{2}b = \frac{1}{2}(24\text{ m}) = 12\text{ m}$ $a = 10\text{ m}$ Find: hypotenuse : $x = ?$

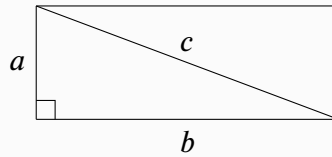
Use the Pythagorean theorem:

$$\begin{aligned} c^2 &= a^2 + b^2 && \text{|substitute: } c = x, a = a \text{ and } b = y \\ x^2 &= a^2 + y^2 && \text{|substitute values: } a = 10\text{ m} \text{ and } y = 12\text{ m} \\ x^2 &= (10\text{ m})^2 + (12\text{ m})^2 \\ x^2 &= 100\text{ m}^2 + 144\text{ m}^2 \\ x^2 &= 244\text{ m}^2 && \text{|take the square root of each side} \\ \sqrt{x^2} &= \sqrt{244\text{ m}^2} \\ x &= 15.6\text{ m} \end{aligned}$$

Area of Right Angled Triangle

The area of a triangle is equal to one half the base multiplied by the corresponding height: $A = \frac{bh}{2}$

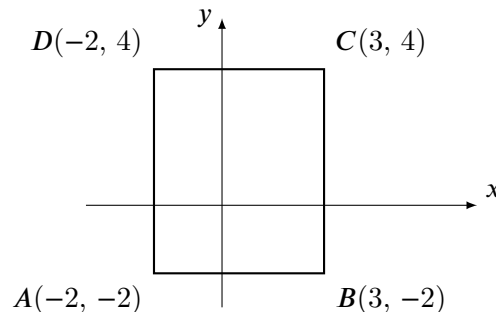
Example 3: Find the length of the diagonal of a rectangle that has width $a = 3$ and length $b = 4$.



The diagonal of a rectangle is the hypotenuse of a right-angle triangle. Use the Pythagorean Theorem.

$$\begin{aligned}c^2 &= a^2 + b^2 \\c^2 &= 3^2 + 4^2 = 9 + 16 = 25 \\ \sqrt{c^2} &= \sqrt{25} \\ c &= 5 \quad \text{The length of the diagonal is } c = 5.\end{aligned}$$

Challenge 2: What is the length BD shown in the figure? 🧮 🟢



Points A , B , and D form the right-angle triangle with hypotenuse BD .

The length AB equals to change in x , from -2 in $A(\underline{-2}, -2)$ to 3 in $B(\underline{3}, -2)$, since points A and B have a fixed y coordinate.

$$AB = \Delta x = 3 - (-2) = 3 + 2 = 5$$

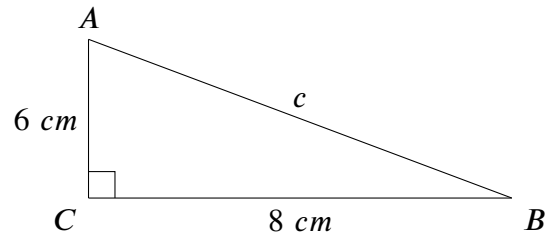
The length AD equals to change in y , from -2 in $A(-2, \underline{-2})$ to 4 in $B(-2, \underline{4})$, since points A and D have a fixed x coordinate.

$$AD = \Delta y = 4 - (-2) = 4 + 2 = 6$$

Use the Pythagorean theorem to find length BD , which is a hypotenuse of the right-angle triangle BAD .

$$\begin{aligned}BD^2 &= 5^2 + 6^2 = 25 + 36 \\ BD^2 &= 61 \\ \sqrt{BD^2} &= \sqrt{61} \\ BD &\approx 7.8\end{aligned}$$

Challenge 3: Find the area of a square whose perimeter is the same as the perimeter of the triangle shown below. 😊



To find the perimeter of the triangle we need to find the hypotenuse, side c , first.

$$P = a + b + c$$

The hypotenuse c of the triangle is

$$\begin{aligned}c^2 &= (6 \text{ cm})^2 + (8 \text{ cm})^2 \\&= 36 \text{ cm}^2 + 64 \text{ cm}^2 \\&= 100 \text{ cm}^2\end{aligned}$$

Take square root of both sides

$$\begin{aligned}\sqrt{c^2} &= \sqrt{100 \text{ cm}^2} \\c &= 10 \text{ cm}\end{aligned}$$

The perimeter of triangle P is the sum of the lengths of all three sides of the triangle:

$$P = 6 \text{ cm} + 8 \text{ cm} + 10 \text{ cm} = 24 \text{ cm}$$

Perimeters of the triangle and the square are same. If the side of the square is a , then the perimeter of the square is

$$P = 4a$$

Divide by 4 and switch sides

$$a = \frac{P}{4}$$

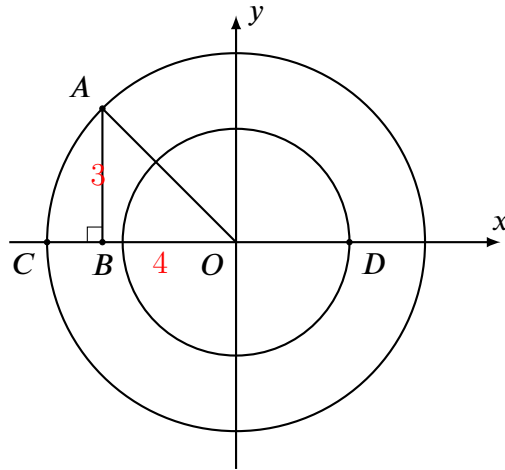
Substitute value for P

$$a = \frac{24 \text{ cm}}{4} = 6 \text{ cm}$$

Finally, the area of the square is

$$A_S = a^2 = (6 \text{ cm})^2 = 36 \text{ cm}^2$$

Challenge 4: Two concentric circles with the center at the origin are shown below. $A(4, 3)$ is on the larger circle and CD is 9, what is the radius of the smaller circle? 🤔



The radius, r_0 of the larger circle is distance AO . To find this distance we use x and y coordinates of point A (right angled triangle $\triangle ABO$). Using Pythagorean theorem we can find the radius of the larger circle, r_0 :

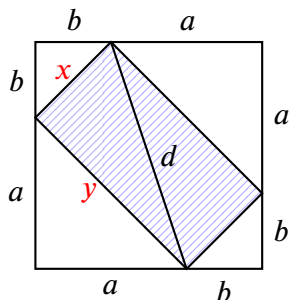
$$r_0^2 = 4^2 + 3^2$$
$$r_0 = \sqrt{16 + 9} = \sqrt{25} = 5$$

Line segment CD is split into CO and OD . Furthermore, $CD = 9$, and CO is the radius of the larger circle, and OD is the radius of the smaller circle

$$CD = CO + OD$$
$$9 = 5 + OD$$
$$OD = 4$$

The radius of the smaller circle is 4.

Challenge 5: A rectangle with the area of $3\sqrt{2}$ is inscribed into a square with the side length of $a + b$, as shown below. Find the length of the rectangles' diagonal d , if $a : b = 2\sqrt{2} : 3$. 😬



Let x and y be the height and the width of the rectangle, respectively. There are four isosceles triangles around the rectangle: two smaller triangles with legs of length b (top left and bottom right corners), and two larger triangles with legs of length a (top right and bottom left corners). Using the Pythagorean Theorem we can write:

$$x^2 = b^2 + b^2$$

$$y^2 = a^2 + a^2$$

$$x^2 = 2b^2$$

$$y^2 = 2a^2$$

$$x = \sqrt{2b^2} = b\sqrt{2}$$

$$y = \sqrt{2a^2} = a\sqrt{2}$$

The area of the rectangle is:

$$A_R = x \cdot y$$

| substitute values $x = b\sqrt{2}$ and $y = a\sqrt{2}$

$$A_R = b\sqrt{2} \cdot a\sqrt{2}$$

$$A_R = 2ab$$

| using the ratio $\frac{a}{b} = \frac{2\sqrt{2}}{3}$, we can write $a = \frac{2\sqrt{2}}{3} \cdot b$

$$A_R = 2 \cdot \left(\frac{2\sqrt{2}}{3} \cdot b \right) \cdot b$$

| simplify and substitute the value $A_R = 3\sqrt{2}$

$$3\sqrt{2} = \frac{4\sqrt{2}}{3}b^2$$

$$| \cdot \frac{4\sqrt{2}}{3}$$

$$3\sqrt{2} \cdot \frac{3}{4\sqrt{2}} = b^2$$

$$b^2 = \frac{9}{4}$$

$$b = \frac{3}{2}$$

Knowing b we can calculate a by using the ratio: $a = \frac{2\sqrt{2}}{3} \cdot b = \frac{2\sqrt{2}}{3} \cdot \frac{3}{2} = \sqrt{2}$. We can now proceed by

calculating $x = b\sqrt{2} = \frac{3}{2} \cdot \sqrt{2} = \frac{3\sqrt{2}}{2}$ and $y = a\sqrt{2} = \sqrt{2} \cdot \sqrt{2} = 2$, and finally calculate the diagonal using the Pythagorean theorem:

$$d^2 = x^2 + y^2$$

$$d = \sqrt{\left(\frac{3\sqrt{2}}{2}\right)^2 + 2^2} = \sqrt{\frac{9 \cdot 2}{4} + 4} = \sqrt{\frac{9}{2} + \frac{8}{2}} = \sqrt{\frac{17}{2}} = \frac{\sqrt{17}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{17}\sqrt{2}}{(\sqrt{2})^2} = \frac{\sqrt{34}}{2}$$